1. (a) Prove that $K_n$ (the $n$-point equilateral space) embeds isometrically into $\ell_p$, for all $1 \leq p \leq \infty$.

(b) Prove that $K_n$ embeds isometrically into $\ell_\infty^{O(\log n)}$.

(c) Show that any embedding of $K_n$ into $\mathbb{R}^1$, incurs distortion at least $n - 1$.

**Bonus** Prove that the number of dimensions needed to embed $K_n$ isometrically into $\ell_2$ is at least $n - 1$.

2. (a) Show that an unweighted graph $K_{1,3}$ (this is basically a 4 point star graph) cannot be isometrically embedded into $\ell_2$.

(b) Find a value $\alpha > 1$ (as big as you can) so that you can show that any embedding of $K_{1,3}$ into Euclidean space requires at least distortion $\alpha$ (Hint: construct an embedding into the plane with the smallest distortion. Try to show this is essentially the best possible way to embed into $\ell_2$).

3. For the next questions, recall the definition of the $\ell_q$-dist $(f)$ we gave in Lecture 1.

(a) Give an embedding of $K_n$ into $\mathbb{R}^1$ with $\ell_1$-dist $(f) = O(\log n)$. Try to improve the embedding to obtain $\ell_1$-dist $(f) = O(\sqrt{\log n})$.

(b) Let $(X, d_X)$, $(Y, d_Y)$ and $(Z, d_Z)$ be any $n$-point metric spaces. Let $f : X \to Y$, and $g : Y \to Z$ be any embeddings. Is it true that $\ell_1$-dist $(f \cdot g) \leq \ell_1$-dist $(f) \cdot \ell_1$-dist $(g)$? Prove that for any $1 \leq q < \infty$, for $(p, s)$ s.t. $\frac{1}{p} + \frac{1}{s} = 1$, it holds $\ell_q$-dist $(f \cdot g) \leq \ell_{qp}$-dist $(f) \cdot \ell_{qs}$-dist $(g)$. Use Hölder’s inequality.

4. Following is a definition of a dimension for general metric spaces.

**Definition 1.1.** Let $(X,d)$ be a metric space. The **doubling constant** of $X$ is the minimal $k$ such that for all $x \in X$, for all $r > 0$, $B(x,r)$ can be covered by at most $k$ balls of radius $r/2$. The **doubling dimension** of $X$ is $\dim(X) = \lceil \log_2(k) \rceil$, where $k$ is the doubling constant of $X$.

Let $X$ be a metric space with doubling dimension $d$.

(a) Let $B(x,r) \subseteq X$ denote a ball of radius $r$ around $x \in X$. Show that for any $\epsilon \leq r/2$, there exists an $\epsilon$-dense $Z \subset B(x,r)$, of size $(\frac{r}{\epsilon})^{O(d)}$.

(b) Use the previous item to show that if $Z \subseteq B(x,r)$ is $\epsilon$-separated, then $|Z| = (\frac{\epsilon}{r})^{O(d)}$. Conclude that there exists an $\epsilon$-net of $B(x,r)$ of size $(\frac{\epsilon}{r})^{O(d)}$. 
